1 Galton-Watson processes

Definition 1.1 *A* Galton-Watson process (GW-process) *is a Markov chain* $\{Z_n : n = 0, 1, \cdots\}$ *on* $\mathbb{N}$ *defined on*

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k},$$

where $\{\xi_{n,k} : n, k \geq 1\}$ *are i.i.d random variables with probability law* $\{p_k\}_{k=0}^{\infty}$ *(*$p_k \geq 0$ *and* $\sum_{k=0}^{\infty} p_k = 1$*).*

Let $f(s)$ be the generating function of ξ defined by

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad s \in [0, 1].$$

Denote by $\{f_n(s)\}_{n\geq 1}$ the iterates of $f(s)$:

$$f_0(s) = s, \quad f_1 = f(s), \quad , f_n(s) = f(f_{n-1}(s))$$

Theorem 1.2 *Let* $m := \mathbb{E}[\xi]$ *and* q *be the smallest root of* $f(t) = t$*, then* $q = 1$ *if* $m \leq 1$ *and* $q < 1$ *if* $m > 1$*.*

Excise 1 *For any* $t \in [0, 1]$*, we have* $f_n(s) \to q$ *as* $n \to \infty$*.*

Theorem 1.3 *As* $n \to \infty$*, we have*

$$\mathbb{P}\{Z_n = 0\} \to q \begin{cases} = 1, & \text{if } m \leq 1; \\ < 1, & \text{if } m > 1. \end{cases}$$

We say GW-process Z_n is *supercritical*, *critical* or *subcritical* if $m > 1$, $m = 1$ or $m < 1$.

Theorem 1.4 *Suppose* $m \in (0, \infty)$*. Let* $\mathscr{F}_n = \sigma(\{Z_0, \cdots, Z_n\})$ *and* $W_n = Z_n/m^n$*. Then* $\{W_n : n = 0, 1, \cdots\}$ *is a* $(\mathscr{F}_n)$*-martingale and there exists a nonnegative random variable* W *such that*

$$W_n \overset{\text{a.s.}}{\to} W, \quad \text{as } n \to \infty.$$

Theorem 1.5 *For the supercritical case $m > 1$, we have*

$$\mathbb{P}\{W = 0\} = q.$$

Moreover, if $\mathbb{E}[|\xi|^2] < \infty$, *then*

$$\mathbb{E}[|W_n - W|] \to 0 \quad \textit{and} \quad \mathbb{E}[W] = 1.$$

Let τ_0 be the extinction time of GW-process Z_n defined by:

$$\tau_0^- := \inf\{n \geq 0 : Z_n = 0\}.$$

Let Z_n^* be the number of individuals in generation n with an infinite line descent. For the k-th particle in generation n, denote by $\xi_{n+1,k}^*$ the number of its offsprings with an infinite line of descent.

Theorem 1.6 *Assume $Z_0 = 1$,*

(1) $\{\xi_{n,k}^* : n \geq 1, k \geq 1\}$ *are i.i.d. random variables and* $\xi^* \leq \xi$. *Moreover,*

$$\mathbb{E}[s^{\xi-\xi^*} t^{\xi^*}] = f(qs + (1-q)t)$$

and

$$\mathbb{E}[s^{Z_n - Z_n^*} t^{Z_n^*}] = f_n(qs + (1-q)t);$$

(2) Conditioned on $\{\tau_0 = \infty\}$, Z_n^* *is also a GW-process with generating function*

$$f^*(s) = \frac{f(q + (1-q)s)}{1-q};$$

(3) Conditioned on $\{\tau_0 < \infty\}$, Z_n^* *is also a GW-process with generating function*

$$f^\circ(s) = \frac{f(qs)}{q};$$

2 Extinction Speed

Theorem 2.1 *If $m = 1$ and $\sigma^2 := \mathrm{Var}(\xi) < \infty$, then as $n \to \infty$*

$$\mathbb{P}\{Z_n > 0\} \sim \frac{2}{n\sigma^2}.$$

Lemma 2.2 *If $m = 1$ and $\sigma^2 < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n}\Big[\frac{1}{1-f_n(t)} - \frac{1}{1-t}\Big] = \frac{\sigma^2}{2}.$$

Lemma 2.3 *Suppose $f(s)$ and $\tilde{f}(s)$ are two generating functions of critical GW-processes with $f'(1) = \tilde{f}'(1) = 1$ and*

$$f''(1) < \tilde{f}''(1) < \infty.$$

Then there exist $k, l \in \mathbb{Z}_+$ such that

$$f_{n+k}(s) \le \tilde{f}_{n+l}(s), \quad s \in [0,1],\ n \ge 0.$$

Theorem 2.4 *If $m = 1$, $\sigma^2 = \infty$ and*

$$f(s) = s + (1-s)^{\alpha+1} L(1-s), \quad \alpha \in (0,1],$$

where $L(s)$ is slowly varying at 0, then

$$\mathbb{P}\{Z_n > 0\} \sim n^{-\frac{1}{\alpha}} L^*(n).$$

Theorem 2.5 (Yaglom's theorem) *If $0 < m < 1$, then*

$$\mathbb{P}\{Z_n = j | Z_n > 0\} \to Q_j,$$

where $\{Q_j : j \ge 1\}$ is a probability law with generating function $g(s)$ is the unique solution to

$$1 - g(f(s)) = m(1 - g(s)).$$

Theorem 2.6 *If $0 < m < 1$, we have*

$$\lim_{n\to\infty} \frac{\mathbb{P}\{Z_n > 0\}}{m^n} = c \geq 0.$$

Moreover, $c > 0$ if and only if

$$\mathbb{E}[\xi \log(1+\xi)] < \infty.$$

Proposition 2.7

$$1 - g(1-s) = sL_2(s),$$

where $L_2(s)$ is a slowly varying at 0.

Theorem 2.8 (Rubin and Vere-Jones (1968)) *Let $f(x)$ be a monotone increasing. If*

$$\lim_{n\to\infty} \frac{f(\lambda\theta_n)}{f(\theta_n)} = \lambda^\alpha, \quad \lambda \in (0,1]$$

for some $\alpha \in \mathbb{R}$ and some sequence $\{\theta\}$ of positive reals tending to 0, as $n \to \infty$, in such way that $\theta_n/\theta_{n+1} < C$, $n \in \mathbb{N}$, $C > 1$, then

$$f(x) = x^\alpha L_3(x),$$

where $L^3(x)$ is slowly varying at 0.